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Strongly $(V^\lambda, A, \Delta^n_{(vm)}, p)$-summable sequence spaces defined by an Orlicz function

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ABSTRACT

We introduce the strongly $(V^\lambda, A, \Delta^n_{(vm)}, p)$-summable sequences and give the relation between the spaces of strongly $(V^\lambda, A, \Delta^n_{(vm)}, p)$-summable sequences and strongly $(V^\lambda, A, \Delta^n_{(vm)}, p)$-summable sequences with respect to an Orlicz function when $A = (a_{ik})$ is an infinite matrix of complex numbers, $\Delta^n_{(vm)}$ is a generalized difference operator and $p = (p_i)$ is a sequence of positive real numbers. Also we give natural relationship between strongly $(V^\lambda, A, \Delta^n_{(vm)}, p)$-convergence with respect to an Orlicz function and strongly $S^\lambda(A, \Delta^n_{(vm)})$-statistical convergence.

1. Introduction

Let $\lambda = (\lambda_r)$ be a nondecreasing sequence of positive numbers tending to ∞, and $\lambda_{r+1} \le \lambda_r + 1$, $\lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by $t_r(x) = \lambda_r^{-1} \sum_{i \in I_r} x_i$, where $I_r = [r - \lambda_r + 1, r]$. A sequence $x = (x_i)$ is said to be (V, λ)-summable to a number s if $t_r(x) \to s$ as $r \to \infty$ [1]. If $\lambda_r = r$, then the (V, λ)-summability is reduced to $(C, 1)$-summability. We write

$$[V, \lambda] = \left\{ x = (x_i) : \lim_{r \to \infty} \lambda_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \right\}$$

for sets of sequences $x = (x_i)$ which are strongly (V, λ)-summable to s, that is, $x_i \to s[V, \lambda]$.

Subsequently strongly (V, λ)-summable as well as generalized this kind of summable sequence spaces have been studied by various authors ([2–6] and others).

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [7] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

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An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)(u \geq 0)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \leq KluM(u)$ for all values of u and $l > 1$.

Remark 1. An Orlicz function satisfies the inequality $M(\lambda x) < \lambda M(x)$ for all λ with $0 < \lambda < 1$.

If the convexity of an Orlicz function is replaced by subadditivity, we call it a modulus function introduced by Nakano [8].

The notion of difference sequence spaces was introduced by Kizmaz [9]. The notion was further generalized by Et and Colak [10] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [11], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. Tripathy et al. [12] generalized the above notions and unified these as follows:

Let m, n be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+mi}.$$

Let m, n be non-negative integers and $v = (v_k)$ be a sequence of non-zero scalars. Then for Z, a given sequence space, recently Dutta [13] introduced the following sequence spaces:

$$Z(\Delta_{(vm)}^n) = \{x = (x_k) \in w : (\Delta_{(vm)}^n x_k) \in Z\}, \quad \text{for } Z = \ell_\infty,\ c \text{ and } c_0$$

where $(\Delta_{(vm)}^n x_k) = (\Delta_{(vm)}^{n-1} x_k - \Delta_{(vm)}^{n-1} x_{k-m})$ and $\Delta_{(vm)}^0 x_k = v_k x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_{(vm)}^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} v_{k-mi} x_{k-mi}.$$

In this expansion it is important to note that we take $v_{k-mi} = 0$ and $x_{k-mi} = 0$ for non-positive values of $k - mi$.

Dutta [13] showed that these spaces can be made BK-spaces under the norm

$$\|x\| = \sup_k |\Delta_{(vm)}^n x_k|.$$

For $n = 1$ and $v_k = 1$ for all $k \in N$, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. For $m = 1$ and $v_k = 1$ for all $k \in N$, we get the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. For $m = n = 1$ and $v_k = 1$ for all $k \in N$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

Parasar and Choudhary [14], Güngöret al. [3], Çolak et al. [15], and others used Orlicz functions for defining some new sequence spaces.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$ converges for each i.

The concept of strong (V, λ)-summability was generalized by Bilgin [2], as follows:

$$V^\lambda[A, f] = \left\{ x = (x_i) : \lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} f(|A_i(x) - s|) = 0,\ \text{for some } s \right\}.$$

In the present paper we introduce the strongly $(V^\lambda, A, \Delta_{(mv)}^n, p)$-summable sequences and give the relation between the spaces of strongly $(V^\lambda, A, \Delta_{(mv)}^n, p)$-summable sequences and strongly $(V^\lambda, A, \Delta_{(mv)}^n, p)$-summable sequences with respect to an Orlicz function when $A = (a_{ik})$ is an infinite matrix of complex numbers and $p = (p_i)$ is a sequence of positive real numbers. Also we give natural relationship between strongly $(V^\lambda, A, \Delta_{(mv)}^n, p)$-convergence with respect to an Orlicz function and strongly $S^\lambda(A, \Delta_{(mv)}^n)$-statistical convergence.

The following inequality will be used throughout the paper:

$$|a_i + b_i|^{p_i} \leq T\{|a_i|^{p_i} + |b_i|^{p_i}\} \tag{1}$$

here a_i and b_i are complex numbers, $T = \max\{1, 2^{H-1}\}$, and $H = \sup p_i < \infty$.

2. Strongly $(V^\lambda, A, \Delta^n_{(mv)}, p)$-summable sequences

Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $p = (p_i)$ be a bounded sequence of positive real numbers $(0 < h = \inf p_i \le p_i \le \sup p_i = H < \infty)$, and M be an Orlicz function. We define

$$V^\lambda[A, \Delta^n_{(mv)}, p, M] = \left\{ x : \lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x) - s|}{\rho} \right) \right]^{p_i} = 0, \text{ for some } s \text{ and } \rho > 0 \right\},$$

$$V_0^\lambda[A, \Delta^n_{(mv)}, p, M] = \left\{ x : \lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x)|}{\rho} \right) \right]^{p_i} = 0, \text{ for some } \rho > 0 \right\},$$

$$V_\infty^\lambda[A, \Delta^n_{(mv)}, p, M] = \left\{ x : \sup_r \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x)|}{\rho} \right) \right]^{p_i} < \infty, \text{ for some } \rho > 0 \right\},$$

where $\Delta^n_{(mv)} A_i(x) = \Delta^{n-1}_{(mv)} A_i(x) - \Delta^{n-1}_{(mv)} A_{i-m}(x) = \sum_{k=1}^\infty \Delta^{n-1}_{(mv)} a_{ik} x_k - \sum_{k=1}^\infty \Delta^{n-1}_{(mv)} a_{i-m,k} x_k$, where $\Delta^n_{(vm)} a_{jk} = \sum_{i=0}^n (-1)^i \binom{n}{i} v_k a_{j-mi,k}$ and we take $a_{j-mi,k} = 0$ for non-positive values of $j - mi$. (*e.g.*, $\Delta^2_{(3)} v_1 a_{11} = v_1 a_{11} - 2v_1 a_{-2,1} + v_1 a_{-5,1} = v_1 a_{11}$, $\Delta^2_{(3)} v_1 a_{71} = v_1 a_{71} - 2v_1 a_{41} + v_1 a_{11}$ *etc.*)

If we take $n = 0$ in the difference operator $\Delta^n_{(mv)}$, we get the spaces $V^\lambda[A, p, M]$, $V_0^\lambda[A, p, M]$ and $V_\infty^\lambda[A, p, M]$.

A sequence $x = (x_k)$ is said to be strongly $(V^\lambda, A, \Delta^n_{(mv)}, p)$-convergent to a number s with respect to an Orlicz function if there is a complex number s such that $x \in V^\lambda[A, \Delta^n_{(mv)}, p, M]$. If x is strongly $(V^\lambda, A, \Delta^n_{(mv)}, p)$-convergent to the value s with respect to a modulus M, then we write $x_i \to s$ $(V^\lambda[A, p, \Delta^n_{(mv)}, M])$.

Throughout the paper μ will denote one of the notations 0, 1 or ∞.

When $M(x) = x$, for all $x \in [0, \infty)$ then we write the spaces $V_\mu^\lambda[A, \Delta^n_{(mv)}, p]$ in place of $V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M]$. If $p_i = 1$ for all i, $V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M]$ reduce to $V_\mu^\lambda[A, \Delta^n_{(mv)}, M]$.

We will write $f \simeq g$ for non-negative functions f and g whenever $C_1 f \le g \le C_2 f$ for some $C_j > 0$, $j = 1, 2$.

In this section we examine some topological properties of $V^\lambda[A, \Delta^n_{(mv)}, p, M]$ spaces and investigate some inclusion relations between these spaces.

Theorem 2.1. *Let M be an Orlicz function and X denotes anyone of the spaces $V^\lambda[A, \Delta^n_{(mv)}, p, M]$, $V_0^\lambda[A, \Delta^n_{(mv)}, p, M]$ or $V_\infty^\lambda[A, \Delta^n_{(mv)}, p, M]$. Then X is a linear space over the complex field C.*

Proof. We give the proof only for $V_0^\lambda[A, \Delta^n_{(mv)}, p, M]$ and for other two spaces it will follow on applying similar arguments.

Let $x, y \in V_0^\lambda[A, \Delta^n_{(mv)}, p, M]$, and $\alpha, \beta \in C$. Then there exist some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x)|}{\rho_1} \right) \right]^{p_i} = 0$$

and

$$\lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(y)|}{\rho_2} \right) \right]^{p_i} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Then we have

$$\begin{aligned} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(\alpha x + \beta y)|}{\rho_3} \right) \right]^{p_i} &\le \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(\alpha x)|}{\rho_3} + \frac{|\Delta^n_{(mv)} A_i(\beta y)|}{\rho_3} \right) \right]^{p_i} \\ &\le T \left\{ \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x)|}{\rho_1} \right) \right]^{p_i} + \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(y)|}{\rho_2} \right) \right]^{p_i} \right\}. \end{aligned}$$

This implies that

$$\lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(\alpha x + \beta y)|}{\rho_3} \right) \right]^{p_i} = 0.$$

This proves that $V_0^\lambda[A, \Delta^n_{(mv)}, p, M]$ is linear. □

Theorem 2.2. *If M be any Orlicz function, then the inclusions $V_0^\lambda[A, \Delta^n_{(mv)}, p, M] \subset V^\lambda[A, \Delta^n_{(mv)}, p, M] \subset V_\infty^\lambda[A, \Delta^n_{(mv)}, p, M]$ hold.*

Proof. The inclusion $V_0^\lambda[A, \Delta^n_{(mv)}, p, M] \subset V^\lambda[A, \Delta^n_{(mv)}, p, M]$ is obvious. Now let $x \in V^\lambda[A, \Delta^n_{(mv)}, p, M]$ such that $x_i \to s(V^\lambda[A, \Delta^n_{(mv)}, p, M])$. Then there exists some $\rho > 0$ such that

$$\lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)}A_i(x) - s|}{\rho}\right)\right]^{p_i} = 0.$$

On taking $\rho_1 = 2\rho$, we have

$$\begin{aligned}\lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)}A_i(x)|}{\rho_1}\right)\right]^{p_i} &= \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)}A_i(x) - s + s|}{2\rho}\right)\right]^{p_i} \\ &\le T\left\{\lambda_r^{-1} \sum_{i\in I_r} \left[\frac{1}{2}M\left(\frac{|\Delta^n_{(mv)}A_i(x) - s|}{\rho}\right)\right]^{p_i} + \lambda_r^{-1} \sum_{i\in I_r} \left[\frac{1}{2}M\left(\frac{|s|}{\rho}\right)\right]^{p_i}\right\} \\ &\le T\left\{\lambda_r^{-1} \sum_{i\in I_r} \left[\frac{1}{2}M\left(\frac{|\Delta^n_{(mv)}A_i(x) - s|}{\rho}\right)\right]^{p_i} + \max\left(1, \left[\frac{1}{2}M\left(\frac{|s|}{\rho}\right)\right]^{H}\right)\right\}.\end{aligned}$$

Hence $x \in V_\infty^\lambda[A, \Delta^n_{(mv)}, p, M]$. This completes the proof. □

The proof of the following corollary is a consequence of Theorem 2.2.

Corollary 2.3. *$V_0^\lambda[A, \Delta^n_{(mv)}, p, M]$ and $V^\lambda[A, \Delta^n_{(mv)}, p, M]$ are nowhere dense subsets of $V_\infty^\lambda[A, \Delta^n_{(mv)}, p, M]$.*

Let X be a sequence space. Then X is called

(i) *Solid (or normal) if $(\alpha_i x_i) \in X$ whenever $(x_i) \in X$ for all sequences (α_i) of scalars with $|\alpha_i| \le 1$; for all $i \in N$;*
(ii) *Monotone provided X contains the canonical preimages of all its step spaces.*

If X is normal, then X is monotone.

Theorem 2.4. *The sequence spaces $V_0^\lambda[A, \Delta^n_{(mv)}, p, M]$ and $V_\infty^\lambda[A, \Delta^n_{(mv)}, p, M]$ are solid and hence monotone.*

Proof. Let $\alpha = (\alpha_i)$ be sequence of scalars such that $|\alpha_i| \le 1$; for all $i \in N$. Since M is monotone, we get for some $\rho > 0$

$$\begin{aligned}\lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)}A_i(\alpha x)|}{\rho}\right)\right]^{p_i} &\le \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\sup|\alpha_i|\frac{|\Delta^n_{(mv)}A_i(x)|}{\rho}\right)\right]^{p_i} \\ &\le \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)}A_i(x)|}{\rho}\right)\right]^{p_i},\end{aligned}$$

which leads us to the desired result. □

Theorem 2.5. *Let M_1 and M_2 be two Orlicz functions. Then we have*

$$V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_1] \cap V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_2] \subseteq V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_1 + M_2]$$

Proof. Proof is easy, so omitted. □

Theorem 2.6. *Let M_1 and M_2 be two Orlicz functions such that $M_1 \simeq M_2$. Then we have*

$$V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_1] = V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_2].$$

Now we give relation between strongly $(V^\lambda, A, \Delta^n_{(mv)}, p)$-convergence and strongly $(V^\lambda, A, \Delta^n_{(mv)}, p)$-convergence with respect to an Orlicz function.

Theorem 2.7. *Let the Orlicz functions M_1 and M_2 satisfy the Δ_2-condition. Then*

(i) $V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_1] \subseteq V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_2 \circ M_1]$.
(ii) $V_\mu^\lambda[A, \Delta^n_{(mv)}, p] \subseteq V_\mu^\lambda[A, \Delta^n_{(mv)}, p, M_2]$.

Proof. (i) We consider only case $V_0^\lambda[A, \Delta^n_{(mv)}, p, M_1] \subseteq V_0^\lambda[A, \Delta^n_{(mv)}, p, M_2 \circ M_1]$. Proof of other two cases follow similarly.

Let $x \in V_0^\lambda[A, \Delta^n_{(mv)}, p, M_1]$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $M_2(u) < \varepsilon$ for $0 \le u \le \lambda$. Write $y_i = M_1(\frac{|\Delta^n_{(mv)}A_i(x)|}{\rho})$ and consider

$$\lambda_r^{-1} \sum_{i\in I_r} [M_2(y_i)]^{p_i} = \lambda_r^{-1} \sum_1 [M_2(y_i)]^{p_i} + \lambda_r^{-1} \sum_2 [M_2(y_i)]^{p_i}$$

where the first summation is over $y_i \le \delta$ and the second summation is over $y_i > \delta$. Since M_2 is continuous, we have

$$\lambda_r^{-1} \sum_1 [M_2(y_i)]^{p_i} < \lambda_r^{-1} \sum_{i \in I_r} \max(1, \varepsilon^H).$$

For $y_i > \delta$, we use the fact that

$$y_i < \frac{y_i}{\delta} \le 1 + \left(\frac{y_i}{\delta}\right).$$

Since M_2 is nondecreasing and convex, we have

$$M_2(y_i) < M_2\left(1 + \frac{y_i}{\delta}\right) < \frac{1}{2}M_2(2) + \frac{1}{2}M_2\left(2\frac{y_i}{\delta}\right).$$

Since M_2 satisfies Δ_2-condition, we have

$$M_2(y_i) < \frac{1}{2}K\frac{y_i}{\delta}M_2(2) + \frac{1}{2}K\frac{y_i}{\delta}M_2(2) = Ky_i\delta^{-1}M(2).$$

Hence

$$\lambda_r^{-1} \sum_2 [M_2(y_i)]^{p_i} < \max(1, (K\delta^{-1}M_2(2)))\lambda_r^{-1} \sum_{i \in I_r} (y_i)^{p_i}.$$

Thus we have

$$\lambda_r^{-1} \sum_{i \in I_r} [M_2(y_i)]^{p_i} < \max(1, \varepsilon^H) + \max(1, (K\delta^{-1}M_2(2)))\lambda_r^{-1} \sum_{i \in I_r} (y_i)^{p_i}.$$

Therefore $x \in V_0^\lambda[A, \Delta_{(mv)}^n, p, M_2 \circ M_1]$.

(ii) Taking $M_1(x) = x$, for all $x \in [0, \infty)$, we get the required result. □

Theorem 2.8. *Let M be any Orlicz function. If $\lim_{t\to 0} \frac{M(t)}{t} > 0$ and $\lim_{t\to 0} \frac{M(t)}{t} < \infty$, then $V_\mu^\lambda[A, \Delta_{(mv)}^n, p] = V_\mu^\lambda[A, \Delta_{(mv)}^n, p, M]$.*

Proof. If an Orlicz function M satisfies the given conditions, then we have $M(t) \simeq t$. Hence using Theorem 2.6 we have the result. □

Theorem 2.9. *Let $0 < p_i \le q_i$ for all k and let (q_i/p_i) be bounded.*
Then $V_\mu^\lambda[A, \Delta_{(mv)}^n, q, M] \subset V_\mu^\lambda[A, \Delta_{(mv)}^n, p, M]$.

Proof. If we take $t_i = [M(\frac{\Delta_{(mv)}^n A_i(x)}{\rho})]^{p_i}$ for all i, then using the same technique employed in the proof of Theorem 2 of Nanda [16], it is easy to prove the theorem. □

The following corollary is a consequence of Theorem 2.8.

Corollary 2.10. (i) *If $0 < \inf p_i \le 1$ for all k, then $V_\mu^\lambda[A, \Delta_{(mv)}^n, M] \subset V_\mu^\lambda[A, \Delta_{(mv)}^n, p, M]$.*
(ii) *$1 \le p_i \le \sup p_i = H < \infty$, then $V_\mu^\lambda[A, \Delta_{(mv)}^n, p, M] \subset V_\mu^\lambda[A, \Delta_{(mv)}^n, M]$.*

3. $S^\lambda(A, \Delta_{(vm)}^n)$-statistical convergence

In this section, we introduce natural relationship between strongly $(V^\lambda, A, \Delta_{(vm)}^n, p)$-convergence with respect to an Orlicz function and strongly $S^\lambda(A, \Delta_{(vm)}^n)$-statistical convergence.

In [17], Fast introduced the idea of statistical convergence. This idea was later studied by Fridy [18], Connor [19], Freedman and Sember [20], Salat [21], Savas [4], Schoenberg [22] and the other authors independently.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number l if for every $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n} |K(\varepsilon)| = 0$, where $|K(\varepsilon)|$ denotes the number of elements in $K(\varepsilon) = \{i \in N : |x_i - l| \ge \varepsilon\}$.

The set of statistically convergent sequences is denoted by S.

A sequence $x = (x_i)$ is said to be strongly $S^\lambda(A, \Delta_{(vm)}^n)$-statistically convergent to s if any $\varepsilon > 0$, $\lim_{r\to\infty} \lambda_r^{-1} |KA(\varepsilon)| = 0$, where $|K(\varepsilon)|$ denotes the number of elements in $KA(\varepsilon) = \{i \in I_r : |\Delta_{(vm)}^n A_i(x) - s| \ge \varepsilon\}$.

The set of all strongly $S^\lambda(A, \Delta_{(vm)}^n)$-statistically convergent sequences is denoted by $S^\lambda(A, \Delta_{(vm)}^n)$ (or $S(\lambda, A, \Delta_{(vm)}^n)$).

Now we give the relation between $S^\lambda(A, \Delta_{(vm)}^n)$-statistical convergence and strongly $(V^\lambda, A, \Delta_{(vm)}^n, p)$-convergence with respect to an Orlicz function.

Theorem 3.1. *Let M be an Orlicz function. Then $V^\lambda[A, \Delta_{(vm)}^n, p, M] \subset S^\lambda(A, \Delta_{(vm)}^n)$.*

Proof. Let $x \in V^{\lambda}[A, \Delta^n_{(vm)}, p, M]$. Then there exists some $\rho > 0$ such that

$$\lim_{r\to\infty} \lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x) - s|}{\rho}\right)\right]^{p_i} = 0, \quad \text{for some } s.$$

Now

$$\begin{aligned}
\lambda_r^{-1} \sum_{i\in I_r} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x) - s|}{\rho}\right)\right]^{p_i} &\geq \lambda_r^{-1} \sum_{|\Delta^n_{(mv)} A_i(x)-s|\geq\varepsilon} \left[M\left(\frac{|\Delta^n_{(mv)} A_i(x) - s|}{\rho}\right)\right]^{p_i} \\
&\geq \lambda_r^{-1} \sum_{|\Delta^n_{(mv)} A_i(x)-s|\geq\varepsilon} \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_i} \\
&\geq \lambda_r^{-1} \sum_{|\Delta^n_{(mv)} A_i(x)-s|\geq\varepsilon} \min\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\} \\
&\geq \frac{1}{\lambda_r} |\{i \in I_r : |\Delta^n_{(mv)} A_i(x) - s| \geq \varepsilon\}| \min\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}.
\end{aligned}$$

Hence $x \in S^{\lambda}(A, \Delta^n_{(vm)})$. □

Theorem 3.2. *Let M be a bounded Orlicz function. Then $V^{\lambda}[A, \Delta^n_{(vm)}, p, M] = S^{\lambda}(A, \Delta^n_{(vm)})$.*

Proof. In view of Theorem 3.1, it is sufficient only to show that $S^{\lambda}(A, \Delta^n_{(vm)}) \subset V^{\lambda}[A, \Delta^n_{(vm)}, p, M]$. Let $x \in S^{\lambda}(A, \Delta^n_{(vm)})$. Since M is bounded, so there exists an integer $K > 0$ and $\rho > 0$ such that

$$M\left(\frac{|\Delta^n_{(vm)} A_i(x) - s|}{\rho}\right) \leq K, \quad s \in C.$$

Then for a given $\varepsilon > 0$, we have

$$\begin{aligned}
\lambda_r^{-1} \sum_{i\in I_r} \left[M \frac{|\Delta^n_{(vm)} A_i(x) - s|}{\rho}\right]^{p_i} &= \lambda_r^{-1} \sum_{|\Delta^n_{(vm)} A_i(x)-s|\geq\varepsilon} \left[M \frac{|\Delta^n_{(vm)} A_i(x) - s|}{\rho}\right]^{p_i} \\
&\quad + \lambda_r^{-1} \sum_{|\Delta^n_{(vm)} A_i(x)-s|<\varepsilon} \left[M \frac{|\Delta^n_{(vm)} A_i(x) - s|}{\rho}\right]^{p_i} \\
&\leq K^H \lambda_r^{-1} |\{i \in I_r : |\Delta^n_{(vm)} A_i(x) - s| \geq \varepsilon\}| + \max\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}.
\end{aligned}$$

Taking limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x \in V^{\lambda}[A, \Delta^n_{(vm)}, p, M]$. This completes the proof. □

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